

AD 687289

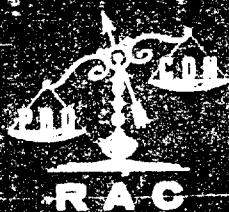
TECHNICAL PAPER

RAC-TP-359

APRIL 1969

RESEARCH ANALYSIS CORPORATION

Bayesian Analysis of the Weibull Process with Unknown Scale and Shape Parameters



D D C
RECEIVED
MAY 13 1969
MAIL ROOM

PREPARED FOR THE
DEPARTMENT OF THE ARMY

CONTRACT NO.
DAGC-17

AD 687289

Reproduced by the
CLEARINGHOUSE
for Federal Scientific & Technical
Information Springfield Va 22151

ADVANCED RESEARCH DEPARTMENT
TECHNICAL PAPER RAC-TP-359
Published April 1969

DISTRIBUTION STATEMENT
This document has been approved for public release and sale; its distribution is unlimited.

Bayesian Analysis of the Weibull Process with Unknown Scale and Shape Parameters

by
Richard M. Soland



Research Analysis Corporation

McLean, Virginia

ACQUISITION FOR

CFSTI

CDC

UNANNOUNCED

JUSTIFICATION

WHITE SECTION ☒

BLUE SECTION ☐

BY

DISTRIBUTION AVAILABILITY CODES

DIST.	AVAIL.	AND/OR SPECIAL
1		

The contents of RAC publications, including the conclusions, represent the views of RAC and should not be considered to have official Department of the Army approval, either expressed or implied.

FOREWORD

This paper presents a contribution to the study of decision making under uncertainty. All decisions are actually made under some degree of uncertainty, and hence it is important to develop models that explicitly take into account the uncertainties present in particular decision situations. Because decisions must be made on the basis of finite information, models are sought that can readily incorporate all available data into the decision-making process. This is what the Bayesian approach attempts to do.

The Weibull process is well known in the reliability field; therefore the Bayesian analysis presented here has potential application to all decisions involving equipments with probabilistic lifetimes. It is an extension of analyses that appeared in RAC-TP-215* and RAC-TP-225.†

Nicholas M. Smith
Head, Advanced Research Department

*R. M. Soland, "Bayesian Analysis of the Weibull Process with Unknown Scale Parameter," RAC-TP-215, Research Analysis Corporation, Aug 66.

†R. M. Soland, "Use of the Weibull Distribution in Bayesian Decision Theory," RAC-TP-225, Research Analysis Corporation, Aug 66.

CONTENTS

Foreword	iii
Abstract	2
1. Introduction	3
2. Prior and Posterior Analyses	4
Definition of the Weibull Process (4)—Likelihood of a Sample (4)— Prior Distribution of $(\hat{\lambda}, \hat{\alpha})$ (5)—Posterior Distribution of $(\hat{\lambda}, \hat{\alpha})$ (6)— Terminal Analysis (6)	
3. Sampling and Preposterior Analysis	7
4. Examples	8
A Reliability Example (9)—A Replacement Example (10)	
References	13
Cited References—Additional References	

**Bayesian Analysis of the
Weibull Process with Unknown Scale
and Shape Parameters**

ABSTRACT

The author previously examined the Weibull process with unknown scale parameter as a model for Bayesian decision making. Here the analysis is extended by treating both the shape and scale parameters as unknown. It is not possible to find a family of continuous joint prior distributions on the two parameters that is closed under sampling; hence a family of prior distributions is used that places continuous distributions on the scale parameter and discrete distributions on the shape parameter. Prior and posterior analyses are examined and seen to be no more difficult than for the case in which only the scale parameter is treated as unknown, but preposterior analysis and determination of optimal sampling plans are considerably more complicated in this case.

Two examples are presented to illustrate the use of the present model. In the first of these it is necessary to make probability statements about the mean life and reliability of a long-life component both before and after life testing. The second example involves determination of the probability distribution of the number of replacement items needed by a group of users during a specified time interval.

1. INTRODUCTION

During the last few years a number of papers have dealt with the Bayesian approach to reliability and maintainability problems (see Refs 1-5). In Ref 6 some desiderata were examined for Bayesian analysis of reliability problems and it was observed that the Weibull distribution possesses the desirable properties of (a) assuming a fairly wide range of shapes depending on the values of the parameters and (b) generating a likelihood function of relatively simple form. In the appendix of Ref 6 we attempted, without success, to find a mathematically tractable continuous joint prior distribution for the scale and shape parameters of the Weibull process that would lead to a posterior distribution of the same form. In Ref 7 a detailed analysis (including prior, posterior, and preposterior analyses) of the Weibull process with unknown scale parameter was performed, i.e., for the situation in which the decision maker, as a matter of policy, assumes the value of the shape parameter to be known.

In many cases, however, it will be desirable to incorporate uncertainty about the shape parameter also, so here the analysis is extended by treating both the shape and scale parameters as unknown. In Sec 2 the Weibull process is defined, the likelihood function is examined, and a family of prior distributions is chosen that places continuous distributions on the scale parameter and discrete distributions on the shape parameter. This family of distributions is closed under sampling and is relatively easy to work with. Prior and posterior analyses are then examined and seen to be no more difficult than for the case in which only the scale parameter is treated as unknown. In Sec 3 preposterior analysis and the determination of optimal sampling plans are examined. These are considerably more difficult than in the case in which only the scale parameter is assumed unknown. It seems that Monte Carlo simulation or a combination of Monte Carlo simulation and numerical integration may be the best way to perform preposterior analysis. Section 4 presents two numerical examples

to illustrate potential uses of the Bayesian approach to reliability in general and of the present model in particular. In the first of these it is necessary to make probability statements about the mean life and reliability of a long-life component both before and after life testing. In such situations it is very probable that few failures will be observed during the life test. The second example involves determination of the probability distribution of the number of replacement items needed by a group of users during a specified time interval. Estimates of such distributions are required both before and after observing the lifetimes of some items in the actual replacement process.

2. PRIOR AND POSTERIOR ANALYSES

Definition of the Weibull Process

A Weibull process is defined as a stochastic process that generates independent random variables $\tilde{x}_1, \dots, \tilde{x}_j, \dots$, with identical densities

$$f_w(x|\lambda, \alpha) = \lambda \alpha x^{\alpha-1} e^{-\lambda x^\alpha} \quad \begin{matrix} 0 \leq x < \infty \\ 0 < \lambda, \alpha < \infty \end{matrix} \quad (1)$$

Here α is the shape parameter, and the quantity $\eta = \lambda^{-1/\alpha}$ is usually called the scale parameter; the parameterization used in Eq (1) is preferred because it "separates" the two parameters (i.e., replaces the factor $\eta^{-\alpha}$ by λ) and thereby simplifies subsequent algebraic manipulations. The distribution function corresponding to the density above is

$$F_w(x|\lambda, \alpha) = 1 - e^{-\lambda x^\alpha} \quad (2)$$

It is henceforth assumed that λ and α are unknown. In practical terms, this is a situation in which a decision maker is willing to assume a Weibull process is generating the independent lifetimes of copies of some particular mechanism or system and wishes to treat the two parameters as unknown.

Likelihood of a Sample

Suppose a censored sample from the Weibull process generates the observations x_1, \dots, x_r and the information $\tilde{x}_{r+1} > x_{r+1}, \dots, \tilde{x}_n > x_n$. Call this evidence z . Such a sample will usually correspond to observing r lifetimes

of x_1, \dots, x_r and observing that $n - r$ mechanisms or systems have operated for times x_{r+1}, \dots, x_n without failing. If the stopping process is noninformative, as defined in Sec 2.3 of Raiffa and Schlaifer,⁸ then the likelihood of this evidence z is

$$L(z|\lambda, \alpha) = \left[\prod_{i=1}^r f_W(x_i|\lambda, \alpha) \right] \left[\prod_{i=r+1}^n [1 - F_W(x_i|\lambda, \alpha)] \right] \\ = \alpha^r \lambda^r (x_1 \dots x_r)^{\alpha-1} \exp \left(-\lambda \sum_{i=1}^n x_i^\alpha \right) \quad (3)$$

Prior Distribution of $(\tilde{\lambda}, \tilde{\alpha})$

When the doublet (λ, α) is to be treated as unknown, as is assumed here, the decision maker ought to place a prior distribution on $(\tilde{\lambda}, \tilde{\alpha})$. It is generally desirable that the prior distribution be a member of a family of distributions that is closed in the sense that the posterior distribution is also a member of this family. The advantage of this is that the same formulas and procedures can be used to calculate expected utility values and the value of information with respect to both the prior and posterior distributions. If sufficient statistics of fixed dimensionality exist, a family of natural conjugate distributions exists and is closed in the above sense (see Ref 8, pp 44-47). Unfortunately, sufficient statistics of fixed dimensionality do not exist for the Weibull process if it is assumed that the shape parameter $\tilde{\alpha}$ may take on any of an infinite number of values.

In the remainder of this paper $\tilde{\alpha}$ is treated as a parameter that may take on any of a finite number of values. Sufficient statistics of fixed dimensionality then exist; the family of natural conjugate distributions then places finite discrete distributions on the specific set of values that $\tilde{\alpha}$ may assume, and, conditional on a particular value of $\tilde{\alpha}$, places a continuous gamma-1 distribution on $\tilde{\lambda}$.

Specifically, this family of distributions allows $\tilde{\alpha}$ to assume any of m values in $(0, \infty)$, and the prior distribution on $(\tilde{\lambda}, \tilde{\alpha})$ is formed as follows: Let

$$p_i = \text{Prob} \{ \tilde{\alpha} = \alpha_i | \quad i = 1, \dots, m. \quad (4)$$

where, of course, $\sum_{i=1}^m p_i = 1$. The conditional density of $\tilde{\lambda}$, given $\tilde{\alpha} = \alpha_i$, call it $f(\lambda|\alpha_i)$, is taken to be the gamma-1 density

$$f(\lambda|\alpha_i) = f_{y_1}(\lambda|r_i', y_i') \frac{y_i'^{r_i' \Gamma(r_i') - 1} e^{-\lambda y_i'}}{\Gamma(r_i')} \quad i = 1, \dots, m. \quad (5)$$

where $0 \leq \lambda < \infty$ and $0 < r_i', y_i' < \infty$. Note that $f(\lambda|\alpha_i)$ depends on α_i only through the dependence of its parameters, r_i' and y_i' , on the index i . Also note that the superscript ' is used to designate parameters of the prior distribution; the superscript " is used similarly for the posterior distribution.

Posterior Distribution of $(\tilde{\lambda}, \tilde{\alpha})$

In the expression for the likelihood of the evidence z , Eq (3), we define the statistics $r, v \equiv \prod_{j=1}^r x_j$, and $y_i \equiv \sum_{j=1}^n x_j^{\alpha_i}$, $i = 1, \dots, m$. The likelihood may then be written as

$$l(z|\lambda, \alpha_i) = \alpha_i^r \lambda^r v^{\alpha_i - 1} e^{-\lambda y_i}.$$

Bayes' theorem then yields a posterior distribution of the same form as the prior distribution with

$$\text{Prob}(\tilde{\alpha} = \alpha_i | z) = p_i' \frac{p_i' \alpha_i^{r_i'} v^{\alpha_i'} y_i'^{r_i' \Gamma(r_i') - 1} \Gamma(r_i') [y_i'^{r_i'} \Gamma(r_i')]}{\sum_{i=1}^m p_i' \alpha_i^{r_i'} v^{\alpha_i'} y_i'^{r_i' \Gamma(r_i') - 1} \Gamma(r_i') [y_i'^{r_i'} \Gamma(r_i')]} \quad (6)$$

$i = 1, \dots, m$, and

$$f(\lambda|\alpha_i, z) = f_{y_1}(\lambda|r_i'', y_i''), \quad (7)$$

where

$$r_i'' = r_i' + r, \quad y_i'' = y_i' + y_i. \quad (8)$$

Note that Eq (6) gives a posterior marginal probability and Eq (7) gives a posterior conditional density.

Terminal Analysis

Suppose the decision maker wishes to choose an act a from a set A of possible acts, and his terminal utility (terminal means that no sample information is to be obtained) for an act a and particular value (λ, α) is $u_i(a; \lambda, \alpha)$. The expected utility of act a is then

$$\begin{aligned} E_{\lambda, \alpha} u_i(a; \tilde{\lambda}, \tilde{\alpha}) &= E_{\tilde{\alpha}} E_{\tilde{\lambda}|\tilde{\alpha}} u_i(a; \tilde{\lambda}, \tilde{\alpha}) \\ &= \sum_{i=1}^m p_i' [E_{\lambda|\alpha_i} u_i(a; \lambda, \alpha_i)] \end{aligned} \quad (9)$$

The notations $E'_{\lambda, \alpha}$, $E'_{\lambda|\alpha_i}$, etc indicate that expectation is taken with respect to the prior distribution of $(\tilde{\lambda}, \tilde{\alpha})$, the conditional prior distribution of $\tilde{\lambda}$, given $\tilde{\alpha} = \alpha_i$, etc. Equation (9) shows that for our choice of prior distribution on $(\tilde{\lambda}, \tilde{\alpha})$ it is no more difficult to compute the expected utility of act a than it is for the case in which α is assumed known and the prior distribution on $\tilde{\lambda}$ is gamma-1 (see Refs 6 and 7).

The decision maker ought to choose an act a' whose expected utility is at least as great as that of all other acts, i.e., $E'_{\lambda, \alpha} u_i(a'; \tilde{\lambda}, \tilde{\alpha}) \geq E'_{\lambda, \alpha} u_i(a; \tilde{\lambda}, \tilde{\alpha})$, $a \in A$. If he desires an indication of how much is at stake if he makes a decision based solely on his prior distribution, he can compute the expected value of perfect information (EVPI):

$$\begin{aligned} \text{EVPI} &= E'_{\lambda, \alpha} \max_a u_i(a; \tilde{\lambda}, \tilde{\alpha}) - E'_{\lambda, \alpha} u_i(a'; \tilde{\lambda}, \tilde{\alpha}) \\ &= \sum_{i=1}^m p_i [E'_{\lambda|\alpha_i} \max_a u_i(a; \tilde{\lambda}, \alpha_i)] - E'_{\lambda, \alpha} u_i(a'; \tilde{\lambda}, \tilde{\alpha}). \end{aligned}$$

Thus the EVPI may also be obtained with no more difficulty than when α is assumed known.

If an experiment e yields evidence z that leads to a posterior distribution with parameters p_i^z , r_i^z , and y_i^z , $i = 1, \dots, m$, the decision maker will compute the expected utility of each act with respect to this posterior distribution. Expression (9) is used with p_i and $E'_{\lambda|\alpha_i}$ replaced by p_i^z and $E'_{\lambda|z, \alpha_i}$.

3. SAMPLING AND PREPOSTERIOR ANALYSIS

If the decision maker is contemplating experimentation he will generally have a utility function $u(e; z; a; \lambda; \alpha)$ defined for each combination of experiment e , outcome z , subsequent action a , and state of nature (λ, α) . Before experiment e is performed the evidence \tilde{z} is a random variable, and so the overall utility of experiment e is

$$u^*(e) = E_{z|e} \max_a E'_{\lambda, \alpha|z} u(e; \tilde{z}; a; \tilde{\lambda}, \tilde{\alpha}). \quad (10)$$

The final step in preposterior analysis is to find an optimal experiment e^* , i.e., one such that

$$u^*(e^*) = \max_e u^*(e).$$

It should be clear from Eq (10) that the distribution of \tilde{z} is a necessary element in the process of preposterior analysis. In our case the evidence \tilde{z} is the vector $(\tilde{r}, \tilde{r}, \tilde{y}_1, \dots, \tilde{y}_m)$. Unfortunately, it does not appear to be possible to obtain the joint distribution of these statistics in any reasonable form.

One way to evaluate $u^*(c)$ for a particular experiment or sampling plan c is to use Monte Carlo simulation to estimate the right-hand side of Eq (10). This can be done by replacing the operator $E_{z|c}$ by the sequence $E_{\lambda, \alpha}, E_{z|c, \lambda, \alpha}$. For particular values of λ and α a sequence of values z_k may be generated via Monte Carlo for the particular experiment c . For each k the quantity $\max_a E_{\lambda, \alpha|z_k} u(c; z_k; a; \tilde{\lambda}, \tilde{\alpha})$ can be readily obtained by terminal analysis so that an estimate of

$$E_{z|c, \lambda, \alpha} \max_a E_{\lambda, \alpha|z} u(c; z; a; \lambda, \alpha) \quad (11)$$

may be obtained. To complete the estimation of $u^*(c)$ the expected value of the quantity in Eq (11) must be taken with respect to the prior distribution of $(\tilde{\lambda}, \tilde{\alpha})$; and this may be approximated by computing a sequence of estimates of the quantity in Eq (11), one for each random draw of a pair (λ, α) from the prior distribution, and using the average of these estimates. Alternatively, the operator $E_{\lambda, \alpha}$ may be replaced by $\sum_{i=1}^m p_i E_{\lambda|z_i}$; one sequence of random draws of λ is then required for each σ_i .

It will not generally be possible to complete preposterior analysis and find an optimal experiment c^* because of the need to estimate $u^*(c)$ numerically; but an appropriate search procedure, together with the Monte Carlo method outlined above, could be used to find a "very good" experiment. Several sampling plans that might be considered in the present context were briefly discussed in Ref 7.

4. EXAMPLES

In this section two small examples are presented to illustrate the potential uses of the Bayesian treatment of a Weibull process with unknown scale and shape parameters. In the first of these it is necessary to make probability statements about the mean life and reliability of a long-life component both before and after life testing. The second example involves estimation of the

probability distribution of the number of replacement items needed by a group of users during a specified future time interval.

A Reliability Example

A certain long-life component was developed for use in a communications satellite. Before a decision was made concerning its use in the satellite, or whether several of them should be used in parallel redundancy, some information was desired about its reliability and mean life. Based on experience with similar components, the design engineers felt that the lifetime distribution of the present model could be adequately represented by a Weibull distribution with shape parameter α less than one. Three possible values of α were chosen: $\alpha_1 = 0.7$, $\alpha_2 = 0.8$, and $\alpha_3 = 0.9$. The prior probabilities decided on were $p_1 = 0.3$, $p_2 = 0.4$, and $p_3 = 0.3$. The engineers felt that the probabilities were 0.5 that such a component would last more than 12,000 hr and 0.2 that one would last more than 41,000 hr. Assuming that these statements were to hold for each of the three possible values of α , and using the fact that

$$P_{\lambda, \alpha_i} [1 - F_W(t|\lambda, \alpha)] = [y_i + (y_i' + t^{\alpha_i})^{-1}]^{-r_i},$$

we numerically solved for the appropriate prior parameters r_i' and y_i' and found $r_1' = 26.12$, $y_1' = 2.665 \times 10^4$, $r_2' = 3.44$, $y_2' = 8.218 \times 10^3$, $r_3' = 1.91$, $y_3' = 1.074 \times 10^4$.

The following statements then follow from the prior distribution on $(\tilde{\lambda}, \tilde{\alpha})$. The probability that such a component will last at least 6 months (12 months) is 0.721 (0.577). The expected mean life is 30,280 hr and the probability is 0.418 that the mean life lies between 20,000 hr and 40,000 hr.

It was decided to get additional information about the lifetime characteristics of this component by performing a life test under simulated operating conditions. Twenty-five items were placed on life test. The first failure occurred after 600 hr and the second occurred after 1500 hr. After each of the failures a new item was substituted for the failed one. Now the life test has been in operation for 2000 hr and we wish to make an interim evaluation of the lifetime characteristics. The sufficient statistics of the data are $r = 2$, $u = (600)(1500) = 9 \times 10^5$, $y_1 = (23)(2000)^{0.7} + (600)^{0.7} + (1400)^{0.7} + (1500)^{0.7} + (500)^{0.7} = 5.192 \times 10^3$, $y_2 = 11.046 \times 10^3$, $y_3 = 23.496 \times 10^3$. Bayes' theorem

yields the following parameters of the posterior distribution: $p_1^* = 0.191$, $p_2^* = 0.430$, $p_3^* = 0.379$, $r_1^* = 28.12$, $y_1^* = 3.184 \times 10^4$, $r_2^* = 5.44$, $y_2^* = 1.926 \times 10^4$, $r_3^* = 3.91$, $y_3^* = 3.434 \times 10^4$.

The posterior distribution on $(\tilde{\lambda}, \tilde{\alpha})$ implies the following statements. The probability that such a component will last at least 6 months (12 months) is 0.790 (0.665). The expected mean life is 37,460 hr and the probability is 0.328 that the mean life lies between 30,000 hr and 50,000 hr. If two such components are placed in active redundancy, the probability is 0.875 that at least one of them will be operating after 12 months. This probability is obtained as

$$1 - E_{\tilde{\lambda}, \tilde{\alpha}} [F_w(t|\lambda, \alpha)]^2$$

for $t = 12$ months.

A Replacement Example

A common problem is the estimation of future demand for replacement parts. One discussion of the application of renewal theory to this problem is found in Ref 9; Goldman¹⁰ and Howard¹¹ have discussed the advantages of using Bayesian methods in such problems. Here an example will be worked out in which it is desired to estimate the probability distribution of the number of replacement items to be demanded by a fleet of users, and the lifetime distribution of the item in question is assumed unknown.

A new type of fuel pump has been developed and placed in each of 10,000 new trucks that the company has recently assembled. We wish to estimate the probability distribution of the number of replacement pumps to be demanded in the next 2 years. We shall treat each new truck as an ordinary renewal process with respect to the replacement of fuel pumps. The characteristics of such a renewal process are determined by the underlying lifetime distribution, i.e., the lifetime distribution of the fuel pumps. We assume this distribution to be a Weibull distribution with (λ, α) unknown. The prior distribution on $(\tilde{\lambda}, \tilde{\alpha})$ has $\alpha_1 = 3.50$, $\alpha_2 = 3.75$, $\alpha_3 = 4.00$, $p_1^* = 0.3$, $p_2^* = 0.35$, $p_3^* = 0.35$, $r_1^* = 12$, $y_1^* = 4.8 \times 10^{17}$, $r_2^* = 6$, $y_2^* = 3.4 \times 10^{18}$, $r_3^* = 3$, $y_3^* = 2.4 \times 10^{19}$. The expected mean life is then 50,660 miles. We shall assume that each truck travels 30,000 miles in 2 years; we could relax this assumption and instead

use the probability distribution of the number of miles traveled (see Ref 9), but this would unnecessarily complicate the present example.

Our method of estimating the required probability distribution is as follows. For particular values λ and α , some results from renewal theory will be used to determine the mean and variance of the number of replacement pumps demanded for one truck. Because the 10,000 trucks are treated as 10,000 independent renewal processes, we will then add the means and variances, i.e., multiply the mean and variance of a single process by 10,000, to determine the mean and variance of the number of replacements, call it \tilde{n} , for the entire fleet. The distribution of \tilde{n} , still conditional on the values λ and α , is approximately normal. To obtain the unconditional distribution function of \tilde{n} , call it $F(n)$, we would take the expectation, with respect to the prior distribution of $(\tilde{\lambda}, \tilde{\alpha})$, of $F(n|\lambda, \alpha)$, the conditional distribution of \tilde{n} , given $\tilde{\lambda} = \lambda$ and $\tilde{\alpha} = \alpha$. Here just the mean and variance of \tilde{n} will be found.

Let $N_o(t|\lambda, \alpha)$ be the number of renewals in the interval $[0, t]$ in an ordinary renewal process in which the underlying lifetime distribution is Weibull with parameters λ and α ($t = 30,000$ in this case). The quantities $E\{N_o(t|\lambda, \alpha)\}$ and variance $\{N_o(t|\lambda, \alpha)\}$ can be found in the tables of Ref 12 for specific values of λ and α . Now let $n_o(t|\lambda, \alpha)$ be the total number of renewals in the interval $[0, t]$ in 10,000 such renewal processes. Then the quantities $E\{n_o^j(t|\lambda, \alpha)\}$, $j = 1, 2$, may be easily obtained from the moments of $N_o(t|\lambda, \alpha)$. Finally we must compute $E_{\tilde{\lambda}, \tilde{\alpha}}\{E\{n_o^j(t|\lambda, \alpha)\}\}$, $j = 1, 2$, and we shall approximate these quantities by

$$\sum_{i=1}^3 p_i^*(0.1) \sum_u E\{N_o^j(t|\lambda_{ui}, \alpha_i)\}, \quad j = 1, 2.$$

where u takes the values 0.05(0.10) 0.95 and λ_{ui} is the u th fractile of the conditional prior distribution of $\tilde{\lambda}$, given $\tilde{\alpha} = \alpha_i$. This distribution is gamma-1, so the appropriate fractiles may be found in the tables of Thom.¹³

The computations implied in the foregoing yield 2680 and 950 for the mean and standard deviation of the number of replacements for the fleet of 10,000 trucks.

Suppose now that several years have passed and we wish to estimate the probability distribution of the number of replacement pumps to be demanded

in the forthcoming 2 years. Suppose that thorough maintenance records are available for 50 of the trucks so that from these data it is possible to update the probability distribution of $(\tilde{\lambda}, \tilde{\alpha})$; the sufficient statistics are $r = 14$, $v = 3.5 \times 10^{48}$, $y_1 = 5.2 \times 10^{17}$, $y_2 = 6.4 \times 10^{18}$, $y_3 = 9.8 \times 10^{19}$. The posterior distribution then has $p_1^* = 0.984$, $p_2^* = 0.016$, $p_3^* = \text{nil}$, and the posterior expected mean life is 49,380 miles.

To simplify matters it will again be assumed that each truck travels 30,000 miles in 2 years and also that all 10,000 trucks operate during this time interval; both assumptions could be relaxed. The computational procedure is the same as the one used previously, except that, because the trucks are several years old, the pump replacements will be treated for each truck as being generated by an equilibrium renewal process instead of an ordinary renewal process. The computations yield 8100 and 440 for the mean and standard deviation of the total number of replacement pumps for the fleet of 10,000 trucks.

REFERENCES

CITED REFERENCES

1. S. K. Bhattacharya, "Bayesian Approach to Life Testing and Reliability Estimation," Journal of the American Statistical Association, 62: 48-62 (1967).
2. A. M. Breipohl, R. R. Prairie, and W. J. Zimmer, "A Consideration of the Bayesian Approach in Reliability Evaluation," IEEE Transactions on Reliability, R-14: 107-113 (1965).
3. M. S. Holla, "Bayesian Estimates of the Reliability Function," Australian Journal of Statistics, 8: 32-35 (1966).
4. R. J. Schulhof and D. L. Lindstrom, "Application of Bayesian Statistics in Reliability," Proceedings of the 1966 Annual Symposium on Reliability, 1966.
5. M. D. Springer and W. E. Thompson, "Bayesian Confidence Limits for the Reliability of Cascade Exponential Subsystems," IEEE Transactions on Reliability, R-16: 86-89 (1967).
6. R. M. Soland, "Use of the Weibull Distribution in Bayesian Decision Theory," RAC-TP-225, Research Analysis Corporation, Aug 66.
7. ———, "Bayesian Analysis of the Weibull Process with Unknown Scale Parameter and Its Application to Acceptance Sampling," IEEE Transactions on Reliability, R-17: 84-90 (1968).
8. H. Raiffa and R. Schlaifer, Applied Statistical Decision Theory, Division of Research, Harvard Graduate School of Business Administration, Boston, 1961.
9. R. M. Soland, "A Renewal Theoretic Approach to the Estimation of Future Demand for Replacement Parts," Operations Research, 16: 36-51 (1968).
10. T. A. Goldman, "Bayesian Supply Policies for Service-Life Parts," Operations Evaluation Group, Rept ORM-36, Mar 63.
11. R. A. Howard, "Prediction of Replacement Demand," in G. Kreweras and G. Morlat (eds), Proceedings of the Third International Conference on Operational Research, Oslo, 1963, Dunod, Paris, 1963.
12. R. M. Soland, "Renewal Functions for Gamma and Weibull Distributions with Increasing Hazard Rate," RAC-TP-329, Research Analysis Corporation, 1968.
13. H. C. S. Thom, "Direct and Inverse Tables of the Gamma Distribution," Rept EDS-2, Environmental Data Service, Silver Spring, Md., Apr 68.

ADDITIONAL REFERENCES

- Briggs, W. G., "Statistical Decision Theory for Logistics Planning," Rept E-1350, Instrumentation Laboratory, Massachusetts Institute of Technology, Cambridge, Mass., May 63.
- Martel, R. J., "Optimum Bayes Strategies in Truncated Life Testing," paper presented at the 27th National Meeting of the Operations Research Society of America, Boston, May 65.

DOCUMENT CONTROL DATA - R&D		
<small>(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)</small>		
1. ORIGINATING ACTIVITY (Corporate author) Research Analysis Corporation McLean, Virginia 22101		7a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP
3. REPORT TITLE BAYESIAN ANALYSIS OF THE WEIBULL PROCESS WITH UNKNOWN SCALE AND SHAPE PARAMETERS		
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Technical Paper		
5. AUTHOR(S) (First name, middle initial, last name) Richard M. Soland		
6. REPORT DATE April 1969	7a. TOTAL NO. OF PAGES 13	7b. NO. OF REFS 15
8a. CONTRACT OR GRANT NO. DAHC19-69-C-0017 b. PROJECT NO. 009.204 c. d.		9a. ORIGINATOR'S REPORT NUMBER(S) RAC-TP-359 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
10. DISTRIBUTION STATEMENT This document has been approved for public release and sale; its distribution is unlimited.		
11. SUPPLEMENTARY NOTES		12. SPONSORING MILITARY ACTIVITY OCDR, DA, Washington, D. C.
13. ABSTRACT The author previously examined the Weibull process with unknown scale parameter as a model for Bayesian decision making. Here the analysis is extended by treating both the shape and scale parameters as unknown. It is not possible to find a family of continuous joint prior distributions on the two parameters that is closed under sampling, hence a family of prior distributions is used that places continuous distributions on the scale parameter and discrete distributions on the shape parameter. Prior and posterior analyses are examined and seen to be no more difficult than for the case in which only the scale parameter is treated as unknown, but preposterior analysis and determination of optimal sampling plans are considerably more complicated in this case. Two examples are presented to illustrate the use of the present model. In the first of these it is necessary to make probability statements about the mean life and reliability of a long-life component both before and after life testing. The second example involves determination of the probability distribution of the number of replacement items needed by a group of users during a specified time interval.		

DD FORM 1473
1 NOV 61

Bayesian analysis
Weibull distribution
Scale and shape parameters unknown

[illegible]